Dr. Z's Math251 Handout #16.4 [Green's Theorem]

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Problem Type 16.4a: Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C P(x,y)\,dx + Q(x,y)\,dy \quad ,$$

where C is a given curve.

Example Problem 16.4a: Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

$$\int_C x^2y^2\,dx + 4xy^3\,dy \quad ,$$

where C is the triangle with vertices $(0,0)$, $(1,3)$, and $(0,3)$.

Steps

1. Set-up Green's Theorem

$$\int_C P(x,y)\,dx + Q(x,y)\,dy$$

$$= \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad ,$$

where D is the region enclosed by C.

Example

1. Here $P = x^2y^2$, $Q = 4xy^3$, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(4xy^3) - \frac{\partial}{\partial y}(x^2y^2)$$

$$= 4y^3 - 2x^2y \quad .$$

We have to evaluate the area integral

$$\int\int_D (4y^3 - 2x^2y^2)\,dA \quad ,$$

where D is the region inside our triangle.

2. Draw D, and write it as a type I or type II region. Set up our area integral as an iterated integral.

2. Our triangle has one side along the y-axis, so it is more convenient to express it as a type II region. The hypotenuse is the line $y = 3x$, that should be written as $x = y/3$, since **now** y is the boss! So

$$D = \{(x, y) \,|\, 0 \le y \le 3\,,\, 0 \le x \le y/3\,\} \quad .$$

The iterated integral is thus:

$$\int_0^3 \int_0^{y/3} (4y^3 - 2x^2y^2)\, dx\, dy \quad .$$

3. Evalute this iterated integral.

3. The inner integral is

$$\int_0^{y/3} (4y^3 - 2x^2y^2)\, dx = 4y^3x - 2y^2\frac{x^3}{3}\Big|_0^{y/3}$$

$$= 4y^3(y/3) - \frac{2}{3}y^2(y/3)^3 = \frac{4}{3}y^4 - \frac{2}{81}y^5 \quad .$$

The whole thing is:

$$\int_0^3 \left[\int_0^{y/3} (4y^3 - 2x^2y^2)\, dx\right] dy \quad ,$$

$$= \int_0^3 (\frac{4}{3}y^4 - \frac{2}{81}y^5)\, dy \quad = \frac{4}{3}\frac{y^5}{5} - \frac{2}{81}\frac{y^6}{6}\Big|_0^3$$

$$= \frac{4}{3}\cdot\frac{3^5}{5} - \frac{2}{81}\cdot\frac{3^6}{6} - 0 = \frac{309}{5} \quad .$$

Ans.: $\frac{309}{5}$.

Problem Type 16.4b :Use Green's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, (Check orientation of the curve before applying the theorem).

$$\mathbf{F}(x, y) = \langle P(x, y)\,, Q(x, y)\rangle \quad ,$$

where C is a certain given closed curve.

Example Problem 16.4b: Use Green's Theorem to evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, (Check orientation of the curve before applying the theorem).

$$\mathbf{F}(x, y) = \langle \sin x + y\,,\, x + \cos^3 y\rangle \quad ,$$

where C consists of the arc of the curve $y = \sin x$ from $(0, 0)$ to $(\pi, 0)$ and the line segment from $(\pi, 0)$ to $(0, 0)$.

Steps

1. Set-up Green's Theorem

$$\int_C P(x, y)\, dx + Q(x, y)\, dy$$

$$= \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad ,$$

where D is the region enclosed by C. Also decide whether the specified description of the curve is in the **positive** direction (**counterclockwise**) or in the **negative** direction (**clockwise**).

Example

1. Here $P = \sin x + y^2$, $Q = x + \cos^3 y$, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(x + \cos^3 y) - \frac{\partial}{\partial y}(\sin x + y^2)$$

$$= 1 - 2y \quad .$$

We have to evaluate the area integral

$$\int\int_D (1 - 2y)\, dA \quad ,$$

where D is the region inside our closed curve. The way C is described is **clockwise** so it is in the negative direction. So at the end, we have to take the **negative** of the result. So multiplying by (-1), we really have to evaluate

$$\int\int_D (2y - 1)\, dA \quad .$$

2. Draw D, and write it as a type I or type II region. Set up our area integral as an iterated integral.

2. Our region has one side along the x-axis, from $x = 0$ to $x = \pi$, and the other part along the curve $y = \sin x$. It is more convenient to express it as a type I region.

$$D = \{(x, y) \,|\, 0 \le x \le \pi\,,\; 0 \le y \le \sin x\,\} \quad .$$

The iterated integral is thus:

$$\int_0^{\pi} \int_0^{\sin x} (2y - 1)\, dy\, dx \quad .$$

3. Evaluate this iterated integral.

3. The inner integral is

$$\int_0^{\sin x} (2y{-}1)\,dy = y^2{-}y\,\Big|_0^{\sin x} = \sin^2 x{-}\sin x \quad .$$

The whole thing is:

$$\int_0^{\pi} \left[\int_0^{\sin x} (2y-1)\,dy \right] dx \quad ,$$

$$= \int_0^{\pi} (\sin^2 x - \sin x)\,dx$$

$$\int_0^{\pi} (\frac{1-\cos 2x}{2} - \sin x)\,dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + \cos x\,\Big|_0^{\pi}$$

$$= (\frac{\pi}{2}{-}\frac{\sin 2\pi}{4}{+}\cos\pi){-}(\frac{0}{2}{-}\frac{\sin(2\cdot 0)}{4}{+}\cos 0)$$

$$= \frac{\pi}{2} - 0 - 1 - (0 - 0 + 1) = \frac{\pi}{2} - 2 \quad .$$

Ans.: $\frac{\pi}{2} - 2$.